Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 9, No. 3, 2017 Article ID IJIM-00861, 8 pages
Research Article
Science and Research Branch (IAU)

Pseudoconvex Multiobjective Continuous-time Problems and Vector Variational Inequalities

E. Khakrah *, A. Razani †‡, M. Oveisiha §

Received Date: 2016-04-23 Revised Date: 2016-11-24 Accepted Date: 2016-12-26

Abstract

In this paper, the concept of pseudoconvexity and quasiconvexity for continuous -time functions are studied and an equivalence condition for pseudoconvexity is obtained. Moreover, under pseudoconvexity assumptions, some relationships between Minty and Stampacchia vector variational inequalities and continuous-time programming problems are presented. Finally, some characterizations of the solution sets of a single-valued continuous-time programming problem are obtained.

Keywords : Multiobjective continuous-time problem; Generalized vector variational inequality; Efficiency; Generalized convexity.

1 Introduction

An optimization problem is characterized by its objective function that is to be minimized or maximized, depending upon the problem and, for a constrained problem, a given set of constraints. Both constrained and unconstrained problems can be considered and formulated as a variational inequality problem. Variational inequality theory was presented by Stampacchia [1] and applied in [2] to study for partial differential equations with applications in mechanics. Variational inequalities are closely related with many problems of nonlinear analysis, such as equilibrium problems, complementarity and fixed point problems, see e.g. [3, 4, 5, 6].

*Department of Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran.

†Corresponding author. razani@sci.ikiu.ac.ir , Tel:+989123079022.

‡Department of Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran.

§Department of Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran.

In 1980, vector variational inequality was introduced by Giannessi [7] in the setting of finite dimensional Euclidean space. Crespi et al. [8] obtained that under pseudoconvexity functions, any solution of Minty vector variational inequality is also a solution of vector optimization problem and the same result can not be extended to quasiconvex functions. In [9], Nobakhtian considered a nonsmooth multiobjective continuous-time problem and obtained some optimality conditions under generalized convexity. Very recently, Ruiz-Garzon et al. [10] studied the relationships between the Minty and the Stampacchia vector variational inequalities and vector continuous-time programming problems under generalized convexity and monotonicity assumptions.
In this paper, motivated by Ruiz-Garzon et al. in [10], we consider Minty and Stampacchia variational inequalities and obtain some relationships between them and multiobjective continuous-time programming problems. The paper is organized as follows: In Section 2, some basic definitions and preliminary results are presented.

Section 3 is devoted to study pseudoconvex and quasiconvex continuous-time functions and relations between them. In Section 4, we show the equivalence of efficient and weak efficient solutions for Minty and Stampacchia variational inequality problems, and solutions of multiobjective continuous-time programming problems under pseudoconvexity condition. Finally, in Section 5, some conclusions are presented, which summarize this work.

2 Preliminaries

Let $I = [a, b]$ be a real interval and $f : I \times \mathbb{R}^n \mapsto \mathbb{R}^p$ be a p-dimensional continuously differentiable function with respect to each of its arguments. For notational convenience, write x for $x(t)$, where $x : I \to \mathbb{R}^n$ is continuous. The partial derivatives of f with respect to x is defined by

$$f_x = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}).$$

In this paper, we consider that X is a nonempty convex subset of the Banach space $C^0[a, b]$ with the norm $\|x\| = \|x\|_\infty$, for all $x \in X$. Consider the following multiobjective continuous-time problem (MCTP):

$$\begin{aligned} min \int_a^b & f(t,x)dt \\ = & (\int_a^b f^1(t,x)dt, \cdots, \int_a^b f^p(t,x)dt), \end{aligned}$$

such that $x \in X$.

Definition 2.1 *[11] A point $y \in X$ is said to be*

1. *an efficient solution of (MCTP), if for all $x \in X$, the following cannot hold*

$$\int_a^b f^i(t,x)dt \leq \int_a^b f^i(t,y)dt,$$

with strict inequality for at least one $i \in P$, that $P = \{1, \cdots, p\}$.

2. *a weak efficient solution of (MCTP), if for all $x \in X$, the following cannot hold*

$$\int_a^b f^i(t,x)dt < \int_a^b f^i(t,y)dt, \quad \forall i \in P.$$

Now, we recall some known concepts of generalized convexity. Let $g : I \times X \mapsto \mathbb{R}$ be a differentiable function.

Definition 2.2 *A functional $\int_a^b g(t,x)dt$ is said to be*

1. *pseudoconvex on X, if for any $x, y \in X$, one has*

$$\begin{aligned} \int_a^b & g_x(t,y)(x-y)dt \geq 0 \\ \Rightarrow & \int_a^b g(t,x)dt \geq \int_a^b g(t,y)dt \end{aligned}$$

2. *quasiconvex on X, if for any $x, y \in X$, one has*

$$\begin{aligned} \int_a^b & g(t,y)dt \leq \int_a^b g(t,x)dt \\ \Rightarrow & \int_a^b g_x(t,x)(y-x)dt \leq 0. \end{aligned}$$

3. *prequasiconvex on X, if for any $x, y \in X$ and $\lambda \in [0, 1]$, one has*

$$\begin{aligned} \int_a^b & g(t, y + \lambda(x-y))dt \\ \leq & \max\{\int_a^b g(t,x)dt, \int_a^b g(t,y)dt\}. \end{aligned}$$

In the next theorem, we present a version of mean-value theorem for integral of differentiable functional. The proof is similar to the standard mean-value theorems.

Theorem 2.1 *Let $g : I \times X \mapsto \mathbb{R}$ be a differentiable function. Then for any $x, y \in X$, there exists $x_0 \in (x, y)$ such that the following inequality holds*

$$\begin{aligned} \int_a^b & g(t,y)dt - \int_a^b g(t,x)dt \\ = & \int_a^b g_x(t,x_0)(y-x)dt \end{aligned}$$

Proof. Set $\varphi : [0, 1] \to \mathbb{R}$ to be a real-valued function defined by

$$\begin{aligned} \varphi(\lambda) = & \int_a^b g(t, x + \lambda(y-x))dt \\ - & \int_a^b g(t,x)dt - \lambda[\int_a^b g(t,y)dt \\ - & \int_a^b g(t,x)dt]. \end{aligned}$$

Applying now Roll's theorem to the function φ and taking into account the chain rule, we can deduce the proof.

Let K be a convex subset of a vector space X. Then a mapping $F : X \rightrightarrows X$ is called a KKM mapping iff for each nonempty finite subset A of K, conv(A)$\subseteq F(A)$, where conv(A) denotes the convex hull of A, and $F(A) = \bigcup\{F(x) : x \in A\}$.

Lemma 2.1 *(see e.g. [12]) Let K be a nonempty and convex subset of a Hausdorff topological vector space X. Suppose that $\Gamma, \widehat{\Gamma} : K \rightrightarrows K$ are two set-valued mappings such* *that the following conditions are satisfied:*

(A1) $\widehat{\Gamma}(x) \subseteq \Gamma(x), \quad \forall x \in K,$
(A2) $\widehat{\Gamma}$ *is a* KKM *map,*
(A3) Γ *is a closed-valued,*
(A4) *there is a nonempty compact convex set* $B \subseteq K$*, such that* $\mathrm{cl}_K(\bigcap_{x\in B}\Gamma(x))$ *is compact.*

Then $\bigcap_{x\in K}\Gamma(x) \neq \phi$.

3 Pseudoconvex and prequasiconvex functional

In this section, we obtain some properties of pseudoconvex and prequasiconvex functional and present the relation between them.

Theorem 3.1 *Let* $\int_a^b g(t,x)dt$ *be a pseudoconvex functional on* X*. Then, it is a prequasiconvex on* X*.*

Proof. Suppose that $\int_a^b g(t,x)dt$ is a pseudoconvex functional on X and $\int_a^b g(t,x)dt$ is not prequasiconvex. Hence, there exist $z \in (x,y)$ such that

$$\begin{aligned}\int_a^b &g(t,z)dt \\ > &\max\{\int_a^b g(t,x)dt, \int_a^b g(t,y)dt\}.\end{aligned}$$

Suppose that s to be a real number such that

$$\begin{aligned}\int_a^b &g(t,z)dt \\ > &s \\ > &\max\{\int_a^b g(t,x)dt, \int_a^b g(t,y)dt\},\end{aligned}$$

and set

$$\begin{aligned}\hat{r} := &\sup\{r \in [0,1] : \\ &\int_a^b g(t,x+r(z-x))dt \leq s\}, \\ \hat{x} := &x + \hat{r}(z-x).\end{aligned}$$

Since g is continuous

$$\int_a^b g(t,\hat{x})dt \leq s \text{ and } \hat{x} \neq z. \tag{3.1}$$

Moreover, from definition of $\hat{r}$ and continuity of g, it follows that

$$\int_a^b g(t,w)dt > s, \ \forall w \in (\hat{x}, z]. \tag{3.2}$$

By Theorem 2.1, there exists $a_0 \in (\hat{x}, z)$ such that

$$\begin{aligned}\int_a^b g(t,z)dt - &\int_a^b g(t,\hat{x})dt \\ = &\int_a^b g_x(t,a_0)(z-\hat{x})dt.\end{aligned}$$

Thus, from (3.1) and (3.2), it follows that

$$\int_a^b g_x(t,a_0)(z-\hat{x})dt > 0. \tag{3.3}$$

As $a_0 \in (\hat{x}, z) \subset (\hat{x}, y)$ and $z \in (\hat{x}, y)$, there exist $\lambda_0, \lambda \in (0,1)$ such that $a_0 = \hat{x} + \lambda_0(y-\hat{x})$ and $z = \hat{x} + \lambda(y-\hat{x})$. Therefore

$$\begin{aligned}y - a_0 = &(1-\lambda_0)(y-\hat{x}), \\ z - \hat{x} = &\lambda(y-\hat{x}).\end{aligned}$$

Hence, from (3.3), it follows that

$$\int_a^b g_x(t,a_0)(y-a_0)dt > 0.$$

From this inequality, pseudoconvexity of $\int_a^b g(t,x)dt$ and (3.2), we can deduce that

$$\begin{aligned}\int_a^b g(t,y)dt \geq &\int_a^b g(t,a_0)dt \\ > &s \\ > &\int_a^b g(t,y)dt,\end{aligned}$$

which is a contradiction.
Now, we present the relation between prequasiconvex and quasiconvex functionals.

Theorem 3.2 *Let* $\int_a^b g(t,x)dt$ *be prequasiconvex on* X*. Then,* $\int_a^b g(t,x)dt$ *is quasiconvex function on* X*.*

Proof. Let $\int_a^b g(t,x)dt$ is a prequasiconvex function on X and $\int_a^b g_x(t,x)(y-x)dt > 0$. Therefore

$$\lim_{h\to 0^+} \frac{\int_a^b [g(t,x+h(y-x)) - g(t,x)]dt}{h} > 0.$$

Thus, there exist $\{h_n\} \downarrow 0$ and $M \in \mathbb{N}$ such that

$$\begin{aligned}\int_a^b &g(t,x+h_n(y-x))dt \\ > &\int_a^b g(t,x)dt,\end{aligned}$$

for all $n \geq M$. By this inequality and prequasiconvexity of $\int_a^b g(t,x)dt$ we get

$$\begin{aligned}\int_a^b &g(t,x)dt \\ < &\int_a^b g(t,x+h_n(y-x))dt \\ \leq &\max\{\int_a^b g(t,x)dt, \int_a^b g(t,y)dt\} \\ = &\int_a^b g(t,y)dt.\end{aligned}$$

Therefore theorem is proved.

Applying Theorems 3.1 and 3.2, we obtain the following result.

Corollary 3.1 *Let* $\int_a^b g(t,x)dt$ *be a pseudoconvex functional on* X*. Then* $\int_a^b g(t,x)dt$ *is quasiconvex on* X*.*

In the following theorem, we obtain an equivalent formulation of pseudoconvexity, that is used in the next section.

Theorem 3.3 *The functional* $\int_a^b g(t,x)dt$ *is pseudoconvex on* X*, if and only if for any* $x, y \in X$*, one has*

$$\begin{aligned} &\int_a^b g_x(t,y)(x-y)dt \geq 0 \\ \Rightarrow &\int_a^b g_x(t,x)(y-x)dt \leq 0. \end{aligned} \tag{3.4}$$

Proof. Suppose that $\int_a^b g(t,x)dt$ is pseudoconvex. Assume by the contradiction that, there exist $x, y \in X$ such that

$$\int_a^b g_x(t,y)(x-y) \geq 0, \tag{3.5}$$

$$\int_a^b g_x(t,x)(y-x)dt > 0. \tag{3.6}$$

By Corollary 3.1, $\int_a^b g(t,x)dt$ is quasiconvex. Therefore by (3.6) and quasiconvexity of $\int_a^b g(t,x)dt$, we obtain

$$\int_a^b g(t,y)dt > \int_a^b g(t,x)dt.$$

Since $\int_a^b g(t,x)dt$ is pseudoconvex, then

$$\int_a^b g_x(t,y)(x-y) < 0,$$

which leads to a contradiction with (3.5).
Conversely, suppose that $x, y \in X$ and

$$\int_a^b g(t,y)dt < \int_a^b g(t,x)dt. \tag{3.7}$$

By Theorem 2.1, there exist $\bar{\lambda} \in (0,1)$ and $x_0 = x + \bar{\lambda}(y-x) \in (x,y)$ such that

$$\begin{aligned} &\int_a^b g(t,y)dt - \int_a^b g(t,x)dt \\ &= \int_a^b g_x(t,x_0)(y-x)dt. \end{aligned} \tag{3.8}$$

Since $x - x_0 = -\bar{\lambda}(y-x)$ and by using (3.7) and (3.8), we deduce that

$$\begin{aligned} &\int_a^b g_x(t,x_0)(x-x_0)dt \\ &= -\bar{\lambda}\int_a^b g_x(t,x_0)(y-x)dt \\ &= -\bar{\lambda}(\int_a^b g(t,y)dt - \int_a^b g(t,x)dt) > 0. \end{aligned}$$

The above inequality and (3.4), yields

$$\int_a^b g_x(t,x)(x_0-x)dt < 0.$$

Therefore

$$\int_a^b g_x(t,x)(y-x)dt < 0.$$

Hence $\int_a^b g(t,x)dt$ is pseudoconvex.

4 (MCTP) and variational inequalities

In this section, we obtain some relationships between Pareto solutions for vector multiobjective continuous-time problem and generalized variational inequalities solutions under pseudoconvexity assumption.
Now, we consider the following vector variational inequalities:
(MVVI): The Minty vector variational inequality finds $\bar{x} \in X$ such that there exists no $x \in X$ satisfying

$$\int_a^b {f^i}_x(t,x)(\bar{x}-x)dt \geq 0, \ \forall i \in P,$$

with strict inequality for at least one $i \in P$.
(MWVVI): The Minty weak vector variational inequality finds $\bar{x} \in X$ such that there exists no $x \in X$ satisfying

$$\int_a^b {f^i}_x(t,x)(\bar{x}-x)dt > 0,$$

for all $i \in P$.

(SVVI): The Stampacchia vector variational inequality finds $\bar{x} \in X$ such that there exists no $x \in X$ satisfying

$$\int_a^b {f^i}_x(t,\bar{x})(x-\bar{x})dt \leq 0,$$

for all $i \in P$, with strict inequality for at least one $i \in P$.
(SWVVI): The Stampacchia weak vector variational inequality finds $\bar{x} \in X$ such that there exists no $x \in X$ satisfying

$$\int_a^b {f^i}_x(t,\bar{x})(x-\bar{x})dt < 0,$$

for all $i \in P$.

Theorem 4.1 *Let $\int_a^b f^i(t,x)dt$ be pseudoconvex functional on X for all $i \in P$. Then $\bar{x} \in X$ is a solution of (MVVI) if and only if it is a efficient solution of (MCTP).*

Proof. Assume that $\bar{x} \in X$ is a solution of (MVVI). Suppose to the contrary that $\bar{x}$ is not an efficient solution of (MCTP). Therefore, there exists $x \in X$ such that

$$\int_a^b f^i(t,x)dt \leq \int_a^b f^i(t,\bar{x})dt, \ \forall i \in P, \quad (4.9)$$

where (4.9) is satisfied as a strict inequality for some $i \in P$. Set

$$x(t) = \bar{x} + t(x-\bar{x}),$$

for all $t \in [0,1]$. By using Theorem 2.1, there exists $t_i \in (0,1)$ such that

$$\begin{aligned} &\int_a^b f^i(t,x)dt - \int_a^b f^i(t,\bar{x})dt \\ &= \int_a^b {f^i}_x(t,\bar{x}+t_i(x-\bar{x}))(x-\bar{x})dt. \end{aligned}$$

From this relation and (4.9), we obtain

$$\int_a^b {f^i}_x(t,\bar{x}+t_i(x-\bar{x}))(x-\bar{x})dt \leq 0,$$

which is satisfied as a strict inequality for at least one $i \in P$. Because $t_i \in (0,1)$ for any $i \in P$, we can choose $t^* \in (0,1)$ such that $t^* < \min\{t_i : i \in P\}$. The above inequality can be rewritten as

$$\begin{aligned} &\int_a^b {f^i}_x(t,x(t_i))(x(t^*)-x(t_i))dt \\ &= (t^*-t_i)\int_a^b {f^i}_x(t,x(t_i))(x-\bar{x})dt \\ &\geq 0. \end{aligned}$$

where is satisfied as a strict inequality for at least one $i \in P$. From Theorem 3.3 and pseudoconvexity of $\int_a^b f^i(t,x)dt$, we can deduce that

$$\int_a^b {f^i}_x(t,x(t^*))(x(t_i)-x(t^*))dt \leq 0,$$

for all $i \in P$, where it is satisfied as a strict inequality for $i \in P$. Therefore

$$\int_a^b {f^i}_x(t,x(t^*))(x-\bar{x})dt \leq 0$$

and

$$\begin{aligned} &\int_a^b {f^i}_x(t,x(t^*))(\bar{x}-x(t^*))dt \\ &= -t^*\int_a^b {f^i}_x(t,x(t^*))(x-\bar{x})dt \\ &\geq 0. \end{aligned}$$

This contradicts the fact $\bar{x}$ is a solution of (MVVI).
Conversely, assume that $\bar{x} \in X$ is an efficient solution of (MCTP). Suppose to the contrary that $\bar{x} \in X$ is not a solution of (MVVI), then there exists $x \in X$ such that

$$\int_a^b {f^i}_x(t,x)(\bar{x}-x)dt \geq 0,$$

for all $i \in P$, which is satisfied as a strict inequality for some $k \in P$. From pseudoconvexity of $\int_a^b f^i(t,x)dt$, we get

$$\int_a^b f^i(t,\bar{x})dt \geq \int_a^b f^i(t,x)dt. \quad (4.10)$$

By using Corollary 3.1, $\int_a^b f^i(t,x)dt$ is quasiconvex, therefore

$$\int_a^b f^k(t,\bar{x})dt > \int_a^b f^k(t,x)dt. \quad (4.11)$$

Relations (4.10) and (4.11) contradicts that $\bar{x}$ is a solution of (MCTP).

Remark 4.1 Theorem 4.1 generalizes and improves [10, Theorem 4] from invex functional to pseudoconvex functional on X. Also, it shows that the other side of Theorem 4 in [10] also holds.

Example 4.1 *Consider the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}^2$, $f = (f^1, f^2)$ defined by $f^1(t,x(t)) = x(t)+\alpha$ and $f^2(t,x(t)) = \alpha x(t)$ with $\alpha > 0$. It can be easily shown that the functional $\int_0^1 f^i(t,x(t))dt$ is pseudoconvex for $i = 1,2$. Suppose that*

$$\int_0^1 {f^1}_x(t,x)(y-x)dt \geq 0.$$

Since,

$$\begin{aligned} &\int_0^1 {f^1}_x(t,x)(y-x)dt \\ &= \int_0^1 f^1(t,y)dt - \int_0^1 f^1(t,x)dt, \end{aligned}$$

Therefore, we obtain

$$\int_0^1 f^1(t,y)dt \geq \int_0^1 f^1(t,x)dt.$$

By a similar way, we can see that $\int_0^1 f^2(t,x)dt$ *is also pseudoconvex.*
Let $x : [0,1] \to \mathbb{R}$ *be defined as* $x(t) = kt, \forall k \in \mathbb{R}^+$*. Then by some computation we can see that* $\bar{x} = 0$ *is a solution of (MVVI) and therefore is a solution of (MCTP).*

In the following Theorem, we establish the weak version of Theorem 4.1. The proof is similar to Theorem 4.1, hence it is omitted.

Theorem 4.2 *Let* $\int_a^b f^i(t,x)dt$ *be pseudoconvex functional on* X *for all* $i \in P$*. Then* $\bar{x} \in X$ *is a solution of (MWVVI) if and only if it is a weak efficient solution of (MCTP).*

Theorem 4.3 *Let* $\int_a^b f^i(t,x)dt$ *be pseudoconvex functional on* X *for all* $i \in P$*. If* $\bar{x} \in X$ *is a solution of (SVVI), then it is a solution of (MVVI) and hence, is a solution of (MCTP).*

Proof. Let $\bar{x} \in X$ be a solution of (SVVI). If $\bar{x} \in X$ is not a solution of (MVVI), then there exists $x \in X$ such that for all $i \in P$

$$\int_a^b {f^i}_x(t,x)(\bar{x} - x)dt \geq 0,$$

with strict inequality for at least one $i \in P$. From Theorem 3.3, we know that $\int_a^b f^i(t,x)dt$ satisfies property (3.4) for all $i \in P$. Therefore

$$\int_a^b {f^i}_x(t,\bar{x})(x - \bar{x})dt \leq 0,$$

with strict inequality for at least one $i \in P$. This contradicts the fact that $\bar{x} \in X$ is a solution of (SVVI).

Remark 4.2 Theorem 4.3, generalized and improved Theorem 2 in [13] from convex functional to pseudoconvex functional and Theorem 2 and Corollary 2 in [10] from invex functional to pseudoconvex functional.

Theorem 4.4 *Let* $\int_a^b f^i(t,x)dt$ *be pseudoconvex functional on* X *for all* $i \in P$*. Then* $\bar{x} \in X$ *is a solution of (MWVVI) if and only if* $\bar{x} \in X$ *is a solution of (SWVVI).*

Proof. By using Theorem 3.3 and a similar way of Theorem 6 in [10], we can deduce the proof.

By using Theorems 4.2 and 4.4 we deduce the following result that generalized and improved Theorem 4 in [13] to pseudoconvex multiobjective continuous-time problems.

Corollary 4.1 *Let* $\int_a^b f^i(t,x)dt$ *for* $i \in P$ *be pseudoconvex functional on* X*. If* $\bar{x} \in X$ *is a solution of (SWVVI), then* $\bar{x}$ *is a weak efficient solution of (MCTP).*

Now, we present an existence result for the solution of (MWVVI) and therefore a weak efficient solution of (MCTP).

Theorem 4.5 *Let* $\int_a^b f^i(t,x)dt$ *be pseudoconvex functional on* X *for all* $i \in P$*. Assume that there are a nonempty compact set* $M \subset X$ *and a nonempty compact convex set* $B \subset X$ *such that for each* $x \in X \setminus M$*, there exists* $y \in B$ *such that*

$$\int_a^b {f^i}_x(t,y)(x - y)dt > 0, \qquad \forall i \in P.$$

Then (MWVVI) has a solution and the set of solutions is compact.

Proof. Define two set-valued mappings $\Gamma, \widehat{\Gamma} : X \rightrightarrows X$ by

$$\begin{aligned} \Gamma(y) \quad &:= \{x \in X : \exists i \in P; \\ &\int_a^b {f^i}_x(t,y)(x-y)dt \leq 0\}, \\ \widehat{\Gamma}(y) \quad &:= \{x \in X : \exists i \in P; \\ &\int_a^b {f^i}_x(t,x)(y-x)dt \geq 0\}, \end{aligned}$$

for each $y \in X$. It is obvious that $\Gamma(x)$ and $\widehat{\Gamma}(x)$ are nonempty. Now, we show that all assumptions of Lemma 2.1 is fulfilled.
(1) $\widehat{\Gamma}$ is a KKM mapping on X. Suppose that $\widehat{\Gamma}$ is not a KKM mapping. Then, there exist $\{y_1, y_2, \cdots, y_m\}$ and $\lambda_j \geq 0$, $j = 1, \cdots, m$ with $\sum_{j=1}^m \lambda_j = 1$ such that

$$y_0 = \sum_{j=1}^m \lambda_j y_j \notin \bigcup \{\widehat{\Gamma}(y_j) : j = 1, \cdots m\}.$$

Therefore it follows that $y_0 \notin \widehat{\Gamma}(y_j)$ for all $j = 1, \cdots, m$, i.e.

$$\int_a^b {f^i}_x(t,y_0)(y_j - y_0)dt < 0,$$

for all $i \in P$, for any $j = 1, \cdots, m$. Moreover

$$\begin{aligned} 0 = & \int_a^b {f^i}_x(t, y_0)(y_0 - y_0)dt \\ = & \sum_{j=1}^m \lambda_j \int_a^b {f^i}_x(t, y_0)(y_j - y_0)dt \\ < & \; 0, \end{aligned}$$

for all $i \in P$, which yields a contradiction.
(2) By the pseudoconvexity of $\int_a^b f^i(t, x)dt$, Theorem 3.3 and definitions of Γ and $\widehat{\Gamma}$, we deduce that $\widehat{\Gamma}(y) \subseteq \Gamma(y)$ and therefore Γ is a KKM mapping.
(3) Γ is closed-valued. Let $\{x_n\} \subset \Gamma(y)$ be a sequence which $x_n \to x_0 \in X$. Therefore for all $n \geq 1$, $\exists i \in P$ such that

$$\int_a^b {f^i}_x(t, y)(x_n - y)dt \leq 0.$$

Hence, there exist $i_0 \in P$ and a subsequence of $\{x_n\}$ such that

$$\int_a^b {f^{i_0}}_x(t, y)(x_{n_j} - y)dt \leq 0,$$

for all $j \geq 1$. Now, by taking the limit as j tends to infinity, it follows that

$$\int_a^b {f^{i_0}}_x(t, y)(x_0 - y)dt \leq 0.$$

Therefore $x_0 \in \Gamma(y)$.
(4) $cl(\bigcap_{x \in B} \Gamma(x))$ is compact because $\bigcap_{x \in B} \Gamma(x) \subset M$.
Therefore all of the conditions of Lemma 2.1 are fulfilled by mapping Γ and

$$\bigcap_{y \in X} \Gamma(y) \neq \phi.$$

Hence there exists x such that for any $y \in X$, $\exists i \in P$

$$\int_a^b {f^i}_x(t, y)(x - y)dt \leq 0.$$

Therefore (MWVVI) has a solution and the solution set should be closed and be contained in the compact set M. This shows that it is compact.
Next, suppose that $\bar{S}$ to be the set of all weakly efficient solutions of (MCTP). Now, we give some characterization of the solution sets of pseudoconvex continuous-time programming problem, i.e.

$$min \int_a^b f(t, x)dt,$$

where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}$.

Theorem 4.6 *Let $\int_a^b f(t, x)dt$ be pseudoconvex functional on X and $\bar{x} \in \bar{S}$. Then, $\bar{S} = S_1 = S_2$, where*

$$\begin{aligned} S_1 = \{x \in X : & \int_a^b f_x(t, x)(\bar{x} - x)dt = 0\}, \\ S_2 = \{x \in X : & \int_a^b f_x(t, x)(\bar{x} - x)dt \geq 0\}. \end{aligned}$$

Proof. Suppose that $x \in \bar{S}$. From $\bar{x} \in \bar{S}$ and Theorem 4.2

$$\int_a^b f_x(t, x)(\bar{x} - x)dt \leq 0. \qquad (4.12)$$

Moreover, by using Theorems 4.2 and 4.4, we deduce that x is solution of (WSVVI) and therefore

$$\int_a^b f_x(t, x)(\bar{x} - x)dt \geq 0. \qquad (4.13)$$

Hence, by relations (4.12) and (4.13), we have

$$\int_a^b f_x(t, x)(\bar{x} - x)dt = 0,$$

i.e. $x \in S_1$. It is trivial that $S_1 \subseteq S_2$. Now, consider that $x \in S_2$. Hence

$$\int_a^b f_x(t, x)(\bar{x} - x)dt \geq 0.$$

Now, pseudoconvexity of $\int_a^b f(t, x)dt$ implies that

$$\int_a^b f(t, \bar{x})dt \geq \int_a^b f(t, x)dt.$$

Because $\bar{x} \in \bar{S}$, it shows that $\int_a^b f(t, \bar{x})dt = \int_a^b f(t, x)dt$ and therefore $x \in \bar{S}$, which completes the proof.

5 Conclusions

In this work, we have studied some relationships between (MVVI), (SVVI) and multiobjective continuous-time programming problems, which extend a lot of results in the literature to the pseudoconvex continuous-time functions. For this objective, the notions of pseudoconvexity and quasiconvexity for continuous-time functions has been investigated. As a consequence of above results, some characterizations of the solution sets of a single-valued continuous-time programming problem has been presented.

References

[1] G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, *C. R. Acad Sci. Paris* 9 (1960) 4413-4416.

[2] P. Hartman, G. Stampacchia, On some non-linear elliptic differential-functional equations,*Acta Math.* 115 (1966) 271-310.

[3] K. C. Broder, Fixed Point Theorems with Applications to Economics and Game Theory, *Cambridge University Press*, Cambridge (1985).

[4] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, *Academic Press*, New York, (1980).

[5] F. Facchinei, J. S. Pang, Finite-Dimensional Variational Inequalities and Complementary Problems, *Springer*, New York, (2003).

[6] R. Moradi, A. Razani, Fixed point theorem for non-self mappings and its applications in the modular space, *Int. J. Industrial Mathematics* 8 (2016) 107-117.

[7] F. Giannessi, Theorems on the alternative quadratic programs and complementarity problems Variational Inequalities and Complementarity Problems, *Theory and applications* 151-186, *Wiley, New York,* (1980).

[8] G. P. Crespi, I. Ginchev, M. Rocca, Some remarks on the Minty vector variational principle, *J. Math. Anal. Appl.* 345 (2008) 165-167.

[9] S. Nobakhtian, Nonsmooth multiobjective continuous-time problems with generalized invexity, *J. Global Optim.* 43 (2009) 593-606.

[10] G. Ruiz-Garzón, R. Osuna-Gómez, A. Rufián-Lizana and B. Hernández-Jimnez, Optimality in continuous-time multiobjective optimization and vector variational-like inequalities, *Top* 23 (2015) 198-219.

[11] C. R. Bector, I. Husain, Duality for multiobjective variational problems, *J. Math. Anal. Appl.* 166 (1992) 214-229.

[12] M. Fakhar, J. Zafarani, Generalized vector equilibrium problems for pseudomonotone bifunctions, *J. Optim. Theory Appl.* 126 (2005) 109-124.

[13] M. H. Kim, Relations between vector continuous-time program and vector variational-type inequality, *J. Appl. Math. Comput.* 16 (2004) 279-287.

Elham Khakrah was born in 1986 in Islamabafgharb, Kermanshah, Iran. She got her M.Sc. in Mathematical Analysis from Isfahan University of Technology in 2012. Now, she is a Ph.D. student in pure Mathematics in Imam Khomeini International University, Qazvin, Iran. Her advisor is Prof. AbdolrahmanRazani. Her interest is to study variationalinequalities, and optimization problems.

Abdolrahman Razani was born in 1969 in Khorram Abad, Iran. He got his Ph.D. in Pure Mathematics (Differential Equations) in 2000 from Tarbiat Modarres University, Tehran, Iran. His interest is to study Nonlinear Analysis, Ordinary Differential Equations and Partial Differential Equations. Now, he is professor at the Department of Mathematics, Imam Khomeini International University, Qazvin, Iran.

Morteza Oveisiha is a faculty member of the Mathematics Department at the Imam Khomeini International University, Qazvin, Iran. His field of specialty includes nonlinear analysis, variational inequalities, and optimization. In 2012, he received his PhD in Mathematics (Analysis) in the Department of Mathematics, University of Isfahan, Isfahan, Iran. In 2005, he completed his MSc degree in Pure Mathematics (Differential Equations) in the Department of Mathematics, Sharif University of Technology, Tehran, Iran. In 2002, he completed his BSc degree in Pure Mathematics from the Sharif University of Technology, Tehran, Iran. He is also a member of Iranian Mathematical Society.