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Use of Fuzzy Numbers for Assessing Problem Solving Skills

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Abstract

The importance of Problem Solving (PS) has been realized for such a long time that in a direct or indirect way affects our daily lives in many ways. Assessment cases appear frequently for PS skills which involve a degree of uncertainty and (or) ambiguity. Fuzzy logic, due to its nature of characterizing such cases with multiple values, offers rich resources for dealing with them. On the other hand, Fuzzy Numbers (FNs) play a fundamental role in fuzzy mathematics, analogous to the role played by the ordinary numbers in classical mathematics. In the present paper we utilize the two simplest forms of them, i.e. the Triangular and Trapezoidal FNs, together with the Centre of Gravity (COG) defuzzification technique as assessment tools for PS skills. Our results are illustrated by three examples in which the assessment outcomes of FNs are validated through their comparison with the corresponding outcomes of assessment methods of the bi-valued and fuzzy logic already tested in author's earlier works.

Keywords : Problem Solving Assessment; Fuzzy Logic (FL); Fuzzy Numbers (FNs); Triangular (TFNs) and Trapezoidal (TpFNs) Fuzzy Numbers; Centre of Gravity (COG) Defuzzification Technique.

1 Introduction

Fuzzy logic (FL), the development of which is based on fuzzy sets theory [17, 18], provides a rich and meaningful addition to the classical logic. Unlike the classical logic, which has only two states, true (1) and false (0), FL deals with truth values which range continuously from 0 to 1. Thus something could be half true 0.5 or very likely true 0.9 or probably not true 0.1, etc. In this way FL allows one to express knowledge in a rule format that is close to a natural language expression and therefore it opens the door to construction of mathematical solutions of computational problems which are imprecisely defined.
The assessment of a systems effectiveness (i.e. of the degree of attainment of its targets) with respect to an action performed within the system (e.g. problem-solving, decision making, learning process, etc) is a very important task. This task enables the correction of the system's weaknesses resulting to the improvement of its general performance. The assessment methods that are commonly used in practice are based on the principles of the bi-valued logic. However, uncertain/ambiguous cases frequently appear in our everyday life, for which a crisp assessment characterization is not the more appropriate. For example, a teacher is frequently not sure about a particular numerical grade characterizing a student's performance.
FL, due to its nature of characterizing the ambiguous cases with multiple values, offers wider and richer resources covering such kind of situa-

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tions. This gave as several times in past the impulse to apply principles of FL for the assessment of human or machine - e.g. Case-Based Reasoning (CBR) systems [12]- skills using as tools the corresponding systems total uncertainty (e.g. see [12] and its relevant references), the Center of Gravity (COG) defuzzification technique [6, 15] as well as two recently developed variations of this technique, i.e. the Triangular [7, 8] and the Trapezoidal [9] Fuzzy Assessment Models, denoted for brevity as TFAM and TpFAM respectively. The above fuzzy methods, although they can be used for individual assessment too [13], they are more appropriate for accessing the overall performance of a group of individuals (or objects) sharing common characteristics; e.g. students, players of a game, CBR systems, etc.
In this paper we shall utilize two of the most commonly used forms of Fuzzy Numbers (FNs) [16], i.e. the Triangular (TFNs) and the Trapezoidal Fuzzy Numbers (TpFNs), together with the COG defuzzification technique as tools for assessing Problem Solving (PS) skills. Notice that, in contrast to the fuzzy assessment methods utilized in our earlier works, this new approach is more appropriate for individual assessment. However, we shall adapt and use it for group assessment too.
The importance of PS has been realised for such a long time that in a direct or indirect way affects our daily lives in many ways. Volumes of research have been written about PS and attempts have been made by many educationists and psychologists to make it accessible to all in various degrees. Mathematics by its nature is a subject whereby PS forms its essence. In an earlier paper [11] we have studied the role of the problem for learning mathematics and we have attempted a review of the evolution of research on PS in mathematics education from its emergency as a self sufficient science at the 1960s until today. Also, in Section 4 of [14], we have applied a general fuzzy framework describing a systems operation in situations characterized by a degree of vagueness and (or) uncertainty, for modelling mathematically the PS process.
The rest of the present paper is organized as follows: In Section 2 we present the notion of a Fuzzy Number (FN), while in Section 3 we present the TFNs, the TpFNs, we define basic arithmetic operations on them and we apply the COG technique to defuzzify them. In Section 4 we describe how one can use the TFNs and TpFNs for assessing PS skills and we discuss the advantages and disadvantages of this approach with respect to the fuzzy assessment methods applied in our earlier works. Finally, Section 5 is devoted to our conclusion and a brief discussion for the perspectives of future research on the subject.

2 Fuzzy Numbers

Basic Definitions

A *Fuzzy Number* (FN) is a special form of fuzzy set in the set R of real numbers. FNs play a fundamental role in fuzzy mathematics, analogous to the role played by the ordinary numbers in classical mathematics.
For a better understanding of the readers which are not familiar to the principles of FL we start from the concept of a fuzzy set, introduced by Zadeh in 1965 [17]

Definition 2.1 *Let U denote the universal set of the discourse. Then a fuzzy set A in U or otherwise a fuzzy subset of U, is a set of ordered pairs of the form $= \{(x,\ m_A(x)) : x \in U\}$, where $m_A : U \to [0,1]$ is the membership function of A. The value $m_A(x)$, called the membership degree (or grade) of x in A, expresses the degree to which x verifies the characteristic property of A. The nearer is the value $m(x)$ to 1, the better x verifies the property of A.*

The definition of the membership function is not unique. It depends on the designer's personal criteria, which are usually empiric or statistical (experimental). However, a necessary condition for the fuzzy set A to represent with credibility the corresponding real situation is that these criteria must be compatible to the common logic.
Obviously each classical (crisp) subset A of U can be considered as a fuzzy subset of U with $m(x) = 1$, if $x \in U$ and $m_A(x) = 0$, if $x \notin U$. Most of the concepts of the crisp sets can be extended, with the help of Definition 2.1, to fuzzy sets. For general facts on fuzzy sets we refer to the book [5]. Before presenting the definition of a FN we give the following three introductory definitions:

Definition 2.2 *A fuzzy set A in U with membership function $y = m(x)$ is said to be normal, if there exists x in U, such that $m(x) = 1$.*[1]

Definition 2.3 *Let A be a fuzzy set in U, and let x be a real number of the interval [0, 1]. Then the x-cut of A, denoted by A^x, is defined to be the crisp set $A^x = \{y \in U : m(y) \geq x\}$.*

Definition 2.4 *A fuzzy set A in* **R** *is said to be convex, if its x-cuts A^x are ordinary closed real intervals, for all x in [0, 1].*

For example, for the fuzzy set A whose membership function's graph is represented in Figure 1, we observe that $A^{0.4} = [5, 8.5] \cup [11, 13]$, which means that A is not a convex fuzzy set.

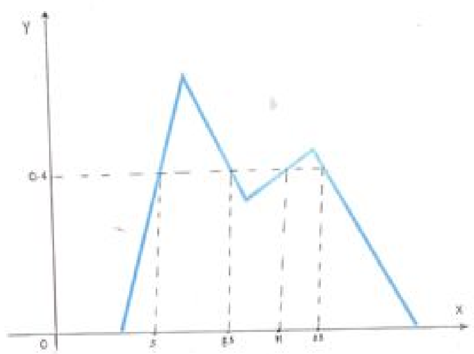

Figure 1: Graph of a non convex fuzzy set

We are ready now to give the definition of a FN:

Definition 2.5 *A fuzzy number is a normal and convex fuzzy set A in* **R** *with a piecewise continuous membership function.*

For example, Figure 2 represents the graph of a FN expressing the fuzzy concept: *"the real number x is approximately equal to 5"*. We observe that the membership function of this FN takes constantly the value 0 outside the interval [0, 10], while its graph in [0, 10] is a parabola.

Since the x-cuts A^x of a FN A are closed real intervals, we can write $A^x = [A_l^x, A_r^x]$ for each x in [0,1], where A_l^x, A_r^x are real numbers depending on x.

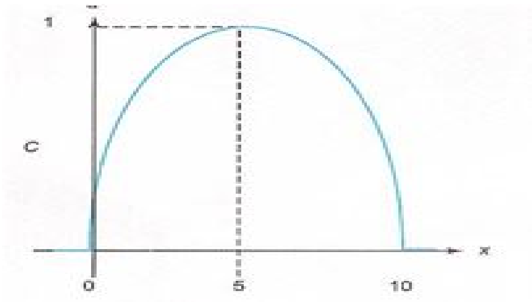

Figure 2: Graphical example of a FN

The following statement defines a *partial order* in the set of FNs:

Definition 2.6 *Given the FNs A and B we write $A \leq B$ (or $\geq$) if, and only if, $A_l^x \leq B_l^x$ and $A_r^x \leq B_r^x$ (or $\geq$) for all x in [0, 1]. Two FNs for which the above relation holds are called comparable, otherwise they are called non comparable.*

2.1 Arithmetic operations on FNs

The basic arithmetic operations on FNs can be defined in two alternative ways, which are equivalent to each other [3]:

1. In terms of their x-cuts, which, as we have already seen, are ordinary closed intervals of **R**. For this, if A and B are given FNs, then an arithmetic operation $*$ between them is defined by $A * B = \sum_{x \in [0,1]} x(A^x * B^x)$, where, for reasons of simplicity, "*" in the second term of the above equation symbolizes the corresponding operation defined on the closed real intervals. Thus, according to this approach, the *Fuzzy Arithmetic* is actually based on the arithmetic of the real intervals.

2. By applying the *Zadeh's extension principle* (see Section 1.4, p.20 of [5]), which provides the means for any function f mapping the crisp set X to the crisp set Y to be generalized so that to map fuzzy subsets of X to fuzzy subsets of Y.

In practice the above two general methods of the fuzzy arithmetic, requiring laborious calculations,

[12.1]There are also alternative methods in use for the symbolic representation of a fuzzy set. In fact, if U is a finite set then A can be written as a symbolic sum of the form $\sum_{x \in U} m_A(x)/x$, if U is a denumerable set then A can be written as a symbolic power series of the form $\sum_{i=0}^{\infty} m_A(x_i)/x_i$, while if U has the power of the continuous then A is usually written as a symbolic integral of the form $\int_U m_A(x)dx$. Notice also that, for reasons of simplicity, many authors identify a fuzzy set with its membership function.

are rarely used in applications, where the utilization of simpler forms of FNs is preferred.
For general facts on FNs we refer to Chapter 3 of the book [10], which is written in Greek language, and also to the books [3, 4].

3 Triangular (TFNs) and Trapezoidal Fuzzy Numbers (TpFNs)

Here we present the two simplest forms of FNs, the Triangular and Trapezoidal FNs.

3.1 Triangular Fuzzy Numbers (TFNs)

Roughly speaking a TFN (a, b, c), with a, b and c real numbers means "approximately equal to b", or that "b lies in the interval (a, c)". We observe that the membership function $y = m(x)$ of (a, b, c) (see Figure 3) takes constantly the value 0, if x is outside the interval [a, c], while its graph in the interval [a, c] is the union of two straight line segments forming a triangle with the OX axis.
Therefore the analytical definition of a TFN is

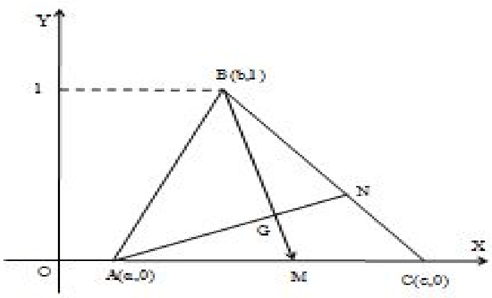

Figure 3: Graph and COG of the TFN (a, b, c)

given as follows:

Definition 3.1 *Let a, b and c be real numbers with $a < b < c$. Then the Triangular Fuzzy Number (TFN) $A = (a,\ b,\ c)$ is the FN with membership function:*

$$y = m(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a, b] \\ \frac{c-x}{c-b}, & x \in [b, c] \\ 0, & x < a \text{ and } x > c \end{cases}$$

Obviously we have that $m(b) = 1$, while b need not be the "middle" of a and c on the real axis.

Using the above definition it is to check that the x-cuts of a TFN $A = (a,\ b,\ c)$ are given by

$$A^x = [a + x(b - a),\ c - x(c - b)] \qquad (3.1)$$

Further, it can be shown [3, 4] that the two general methods of performing the basic arithmetic operations between FNs reported in Section 2 lead to the following simple rules for the addition and subtraction of TFNs:
Let $A = (a,\ b,\ c)$ and $B = (a_1,\ b_1,\ c_1)$ be two TFNs. Then

- The sum $A + B = (a + a_1,\ b + b_1,\ c + c_1)$.

- The difference $A - B = A + (-B) = (a - c_1, b - b_1, c - a_1)$, where $B = (-c_1, -b_1, -a_1)$ is defined to be the opposite of B.

In other words, the opposite of a TFN, as well as the sum and the difference of two TFNs are also TFNs.
On the contrary, the product and the quotient of two TFNs, although they are FNs, they are not always TFNs. However, in the special case where a, b, c, a_1, b_1, c_1 are in $\mathbf{R}^+$, it can be shown that the fuzzy operations of multiplication and division of TFNs can be approximately performed by the rules:

- The product $A.B = (aa_1, bb_1, cc_1)$.

- The quotient $A.B = A.B^{-1} = (\frac{a}{a^{-1}}, \frac{b}{b^{-1}}, \frac{c}{c^{-1}})$, where $B^{-1} = (\frac{1}{a^{-1}}, \frac{1}{b^{-1}}, \frac{1}{c^{-1}})$ is defined to be the inverse of B.

In other words, in R+ the inverse of a TFN, as well as the product and the division of two TFNs can be approximately considered to be TFNs too. Further, one can define the following two scalar operations:

- $k + A = (k + a,\ k + b,\ k + c), k \in \mathbf{R}$.

- $kA = (ka,\ kb,\ kc)$, *if* $k > 0$, and $kA = (kc,\ kb,\ ka)$, *if* $k < 0$ is defined to be the inverse of B.

3.2 Trapezoidal Fuzzy Numbers (TpFNs)

Roughly speaking a TpFN (a, b, c, d) with a, b, c, d in **R** means "approximately in the interval [b, c]". Its membership function $y = m(x)$ is constantly 0 outside the interval [a, d], while its graph in this interval is the union of three straight line segments forming a trapezoid with the X-axis (see Figure 4), Therefore, its analytic definition

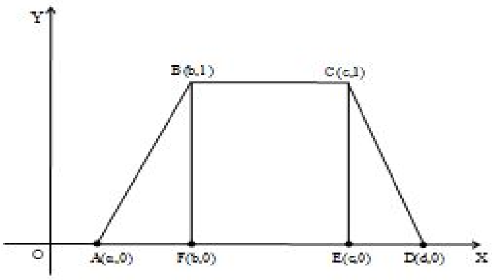

Figure 4: Graph of the TpFN (a, b, c, d)

is given as follows:

Definition 3.2 *Let $a < b < c < d$ be given real numbers. Then the TpFN (a, b, c, d) is the FN with membership function:*

$$y = m(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a, b] \\ x = 1, & x \in [b, c] \\ \frac{d-x}{d-c}, & x \in [c, d] \\ 0, & x < a \text{ and } x > d \end{cases}$$

Obviously the TpFNs are generalizations of TFNs. In fact, the TFN (a, b, d) can be considered as a special case of the TpFN (a, b, c, d) with b=c.

It can be shown that the two general methods for performing the basic arithmetic operations between FNs (see Section 2) lead to the following simple rules for the addition and subtraction of TpFNs:
Let $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ be two TFNs. Then

- The sum $A + B = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3,\ a_4 + b_4)$.
- The difference $A - B = A + (-B) = (a_1 - b_4,\ a_2 - b_3,\ a_3 - b_2,\ a_4 - b_1)$, where $B = (-b_4,\ -b_3,\ -b_2,\ -b_1)$ is defined to be the opposite of B.

In other words, the opposite of a TpFN, as well as the sum and the difference of two TpFNs are also TpFNs.
On the contrary, the product and the quotient of two TFNs, although they are FNs, they are not always TpFNs, apart from some special cases, or in terms of suitable approximating formulas (for more details see [3]).
Further, one can define the following two scalar operations:

- $k+A = (k+a_1,\ k+a_2,\ k+a_3,\ k+a_4), k \in \mathbf{R}$.
- $kA = (ka_1,\ ka_2,\ ka_3,\ ka_4),\ if\ k > 0$, and $kA = (ka_4,\ ka_3,\ ka_2,\ ka_4),\ if\ k < 0$ is defined to be the inverse of B.

We close this section with the following definition, which will be used in the next section for assessing PS skills with the help of TpFNs (or TFNs):

Definition 3.3 *Let A_i, $i = 1,\ 2, ...,\ n$ be TpFNs (or TFNs), where n is a non negative integer, $n \geq 2$. Then the mean value of the above TpFNs (TFNs) is defined to be the TpFN (or TFN) $A = \frac{1}{n}(A_1 + A_2 + ... + A_n)$.*

3.3 Defuzzification of TFNs/TpFNs

Here we use the COG technique for defuzzifying a given TFN/TpFN. We start with the case of TFNs:

Proposition 3.1 *The coordinates (X, Y) of the COG of the graph of the TFN (a, b, c) are calculated by the formulas $X = \frac{a+b+c}{3}$, $Y = \frac{1}{3}$.*

Proof. The graph of the TFN (a, b, c) is the triangle ABC of Figure 3, where A(a, 0), B(b, 1) and C(c, 0). Then, the COG, say G, of ABC is the intersection point of its medians AN and BM, where $N(\frac{b+c}{2},\ \frac{1}{2})$ and $M(\frac{a+c}{2}, 0)$. Therefore the equation of the straight line on which AN lies is $\frac{x-a}{\frac{b+c}{2}-a} = \frac{Y}{\frac{1}{2}}$, or

$$x + (2a - b - c)y = a \tag{3.2}$$

In the same way one finds that the equation of the straight line on which BM lies is

$$2x + (a + c - 2b)y = a + c \tag{3.3}$$

Since $D = \begin{vmatrix} 2 & a+c-2b \\ 1 & 2a-b-c \end{vmatrix} = 3(a-c) \neq 0$, the linear system of (3.2) and (3.3) has a unique solution

with the respect to the variables x and y determining the coordinates of the triangle's COG.
The proof of the Proposition is completed by observing that $D_x = \begin{vmatrix} a+c & a+c-2b \\ a & 2a-b-c \end{vmatrix} = a^2-c^2+ba-bc = (a+c)(a-c)+b(a-c) = (a-c)(a+c+b)$ and $D_y = \begin{vmatrix} 2 & a+c \\ 1 & a \end{vmatrix} = a-c$. Proposition 3.1 can be used as a lemma for the defuzzification of TpFNs. The corresponding result is the following:

Proposition 3.2 *The coordinates (X, Y) of the COG of the graph of the TpFN (a, b, c, d) are calculated by the formulas* $X = \frac{c^2+d^2-a^2-b^2+dc-ba}{3(c+d-a-b)}$, $Y = \frac{2c+d-a-2b}{3(c+d-a-b)}$.

Proof. We divide the trapezoid forming the graph of the TpFN (a, b, c, d) in three parts, two triangles and one rectangle (see Figure 4). The coordinates of the three vertices of the triangle ABE are (a, 0), (b, 1) and (b, 0) respectively, therefore by Proposition 3.1 the COG of this triangle is the point $C_1(\frac{a+2b}{3}, \frac{1}{3})$. Similarly one finds that the COG of the triangle FCD is the point $C_2(\frac{d+2c}{3}, \frac{1}{3})$. Also, it is easy to check that the COG of the rectangle BCFE is the point $C_3(\frac{b+c}{2}, \frac{1}{2})$. Further, the areas of the two triangles are equal to $S_1 = \frac{b-a}{2}$ and $S_2 = \frac{d-c}{2}$ respectively, while the area of the rectangle is equal to $S_3 = c-b$. It is well known then (e.g. see [19]) that the coordinates of the COG of the trapezoid, being the resultant of the COGs $C_i(x_i, y_i)$, for i=1, 2, 3, are calculated by the formulas

$$X = \frac{1}{n}\sum_{i=1}^{3} S_i x_i, \; Y = \frac{1}{n}\sum_{i=1}^{3} S_i y_i \tag{3.4}$$

where $S = S_1+S_2+S_3 = \frac{c+d-b-a}{2}$ is the area of the trapezoid.
The proof of the Proposition is completed by replacing the above found values of S, S_i, x_i and y_i, i = 1, 2, 3, in formulas (3.4) and by performing the corresponding operations.

4 Use of the TFNs/TpFNs for Assessing PS Skills

In this section we use the TFNs /TpFNs and the COG defuzzification technique as tools for assessing PS skills. For this, we consider the following examples:

4.1 Example

Three mathematical modelling problems were given for solution to the students of two different Departments of the School of Management and Economics of the Graduate Technological Educational Institute (T. E. I.) of Western Greece at their common progress exam of the course Mathematics for Economists I. The first problem involved the use of derivatives for maximizing the quantity of water running through a channel, the second one involved the use of square matrices for coding messages and the third problem involved the application of a differential equation with splitting variables for calculating the population of a country. The students of the two Departments achieved the following scores (in a climax from 0 to 10):
First Department (D_1): 100(2 times), 99(3), 98(5), 95(8), 94(7), 93(1), 92 (6), 90(5), 89(3), 88(7), 85(13), 82(6), 80(14), 79(8), 78(6), 76(3), 75(3), 74(3), 73(1), 72(5), 70(4), 68(2), 63(2), 60(3), 59(5), 58(1), 57(2), 56(3), 55(4), 54(2), 53(1), 52(2), 51(2), 50(8), 48(7), 45(8), 42(1), 40(3), 35(1).
Second Department (D_2) : 100(1), 99(2), 98(3), 97(4), 95(9), 92(4), 91(2), 90(3), 88(6), 85(26), 82(18), 80(29), 78(11), 75(32), 70(17), 64(12), 60(16), 58(19), 56(3), 55(6), 50(17), 45(9), 40(6).

4.1.1 Summary of our Previous Research

(for more details see [15]) Calculating the mean values of the above scores, one approximately finds the value 72.44 for D_1 and the value 72.04 for D_2 respectively, showing that D_1 demonstrated a slightly better mean performance than D_2.
Further, in [15] we introduced the set U = A, B, C, D, F of linguistic labels (grades) corresponding to the above scores as follows: A (85-100) = excellent, B (84-75) = very good, C (74-60) = good, D (59-50) = fair and F(¡50)= unsatisfactory.
Then, we calculated the GPA index for each Department by the formula $GPA = \frac{n_D+2n_C+3n_B+4n_A}{n}$ (e.g. cf. [1]), where n_x denotes the number of students of the Department whose scores correspond to the linguistic label (grade) x in U and we found the value GPA = 2.529 for both Departments. The value of GPA measures the quality performance of each Department, because in the above formula higher coefficients are

attached to the higher scores.
Next, in [15] we applied three different fuzzy methods for the student assessment: First, expressing the two Departments as fuzzy sets in U (the membership function was defined by $m(x) = \frac{n_x}{n}$), we calculated the total (possibilistic) uncertainty for each Department after the solution of the given problems by students and we found the values 0.259 for D_1 and 0.934 for D_2 respectively. Thus, the reduction of the initially (i.e. before the solution of the problems) existing uncertainty was much greater for D_1, showing that D_1 demonstrated a considerably better mean performance than D_2. Further, the application of the Center of Gravity (COG) defuzzification technique, as well as of the TpFAM (Trapezoidal) or its equivalent TFAM (Triangular) Fuzzy Assessment Model showed that, in contrast to their mean performance, D_2 demonstrated a slightly better quality performance than D_1 [15].
Finally in [15], the differences appeared in the results obtained by applying the above (five in total) traditional and fuzzy assessment methods, were adequately explained and justified through the nature of each method.

4.1.2 Use of the TFNs

We assign to each linguistic label (grade) of the set U considered in Section *4.1.1* a TFN (denoted for simplicity by the same letter) as follows: A= (85, 92.5, 100), B = (75, 79.5, 84), C = (60, 67, 74), D= (50, 54.5, 59) and F = (0, 24.5, 49). The middle entry of each of the above TFNs is equal to the mean value of the student scores attached to the corresponding linguist label (grade), these scores appearing as the first (left) and third (right) entry respectively of the corresponding TFN. In this way a TFN corresponds to each student assessing his (her) individual performance.
Notice that in [13] an ordered triple of fuzzy linguistic labels was assigned to each student assessing (qualitatively) his/her performance It was shown also [13] that this approach is equivalent to the A. Jones method [2] of assessing a student's knowledge in terms of his (her) fuzzy deviation with respect to the teacher. However, the method with the TFNs presented here is more comprehensive, since it treats the (fuzzy) individual student assessment numerically.
Next, summarizing the students performance in terms of the TFNs defined above we form the Table 1 below as follows: We observe that

Table 1: Students' performance in terms of the TFNs

TFN	D_1	D_2
A	60	60
B	40	90
C	20	45
D	30	45
F	20	15
Total	170	255

in Table 1 we have 170 TFNs representing the progress of the students of D_1 and 255 TFNs representing the progress of the students of D_2. Therefore, it is logical to accept that the overall performance of each Department is given by the corresponding mean value of the above TFNs (see Definition 3.3). For simplifying our symbols, let us denote the above means by the letter of the corresponding Department. Then, making straightforward calculations, one finds that

$D1 = \frac{1}{170}.(60A + 40B + 20C + 30D + 20F) \approx (63.53, 71.74, 83.47)$
and

$D2 = \frac{1}{255}.(60A + 90B + 45C + 45D + 15F) \approx (65.88, 72.63, 79.53)$.

Observing the left entries (63.53 and 65.88 respectively) and the right entries (83.47 and 79.53 respectively) of the TFNs D_1 and D_2 one concludes that the overall performance of the two Departments could be characterized from good (C) to very good (B). It is also of worth to clarify that the middle entries of D_1 and D_2 (71.74 and 72.63 respectively) do not measure the mean performances of the two Departments, which are measured by the mean values 72.44 and 72.04 respectively of the student scores for each Department (see Section *4.1.1*). In fact, since the middle entries of the TFNs A, B, C, D and F were chosen to be equal to the means of the scores assigned to the corresponding linguistic grades, the middle entries of the TFNS D_1 and D_2 are equal to the mean values of these means, i.e. they give a rough approximation only of the performances of the two Departments.
Next, applying formula (3.1) of Section *3.1* one finds that the x-cuts of the two TFNs

are $D_1^x = [63.53 + 8.21x,\ 83.47 - 11.73x]$ and $D_2^x = [65.88 + 6.75x,\ 79.53 - 6.9x]$ respectively. But $63.53 + 8.21x \leq 65.88 + 6.75x \iff 1.46x \leq 2.35 \iff x \leq 1.61$, which is true, since x is in [0, 1]. On the contrary, $83.47 - 11.73x \leq 79.53 - 6.9x \iff 3.94 \leq 4.83x \iff 0.82 \leq x$, which is not true for all the values of x. Therefore, according to Definition 2.6 of Section 2 the TFNs D_1 and D_2 are not comparable. This means that it is not possible to compare the overall performance of the two Departments directly from them.

A good way to overcome this difficulty is to defuzzify the TFNs D_1 and D_2. By Proposition 3.1, the COGs of the triangles forming the graphs of the TFNs D_1 and D_2 have x-coordinates equal to

$$X = \frac{63.53 + 71.74 + 83.47}{3} \approx 72.91$$

and

$$X' = \frac{65.88 + 72.63 + 79.53}{3} \approx 72.68$$

respectively.

Observe now that the GOGs of the graphs of D_1 and D_2 lie in a rectangle with sides of length 100 units on the X-axis (student scores from 0 to 100) and one unit on the Y-axis (normal fuzzy sets). Therefore, the nearer the x-coordinate of the COG to 100, the better the corresponding Department's performance, Thus, since $X > X$, D_1 demonstrates a better overall performance than D_2.

Our next example is suitable for using the TpFNs also for assessing the students' PS skills.

4.2 Example

Six different mathematics teachers train a group of five students of the Upper Secondary Education, who distinguished at their National Mathematical Competition, in order to participate in the International Mathematical Olympiad. In a preparatory test during their training the students ranked with the following scores (from 0-100) by their teachers: S_1 (*Student* 1): 43, 48, 49, 49, 50, 52, S_2: 81, 83. 85, 88, 91, 95, S_3: 76, 82, 89, 95, 95, 98, S_4: 86, 86, 87, 87, 87, 88 and S_5: 35, 40, 44, 52, 59, 62. Assess the student performance with the help of TFNs and TpFNs.

4.2.1 Use of the TFNs

We consider again the TFNs A, B, C, D and F defined in Example *4.1*. Observing the $5*6 = 30$ in total student scores one finds that in the present example we have 14 TFNs equal to A, 4 equal to B, 1 equal to C, 4 equal to D and 7 TFNs equal to F characterizing the student performance. The mean value of the above TFNs (see Definition 3.3) is equal to $M = \frac{1}{30}(14A + 4B + C + 4D + 7F) \approx (60.33, 68.98, 79.63)$. Therefore, the student overall performance lies in the interval [60.33, 79.63], i.e. it could be characterized from good (C) to very good (B). Further, a rough approximation of this performance is given by the score 68.98 (good).

4.2.2 Use of the TpFNs

We assign to each student S_i a TpFN (denoted, for simplicity, with the same letter) as follows: S_1 = (0, 43, 52, 59), S_2 = (75, 81, 95, 100), S_3 = (75, 76, 98, 100), S_4 = (85, 86, 88, 100) and S_5 = (0, 35, 62, 74). Each of the above TpFNs characterizes the individual performance of the corresponding student in the form (a, b, c, d), where a and d are the minimal and maximal scores respectively of the linguistic grades characterizing his/her performance, while b and c are the lower and higher scores respectively assigned to the student by the teachers.

For assessing the overall student performance with the help of the above TpFNs, we calculate the mean value of the TpFNs S_i , i =1, 2, 3, 4, 5 (Definition 3.3), which is equal to the TpFN $\mathbf{S} = \frac{1}{5}\sum_{i=1}^{5} S_i$= (47, 64.2, 79, 86.6).

The TpFN **S** gives the information that the student overall performance lies in the interval [b, c] = [64.2, 79], i.e. it could be characterized from good (C) to very good (B).

Our last example extends Example *4.2* by applying the COG deffuzzification technique for TpFNs in order to compare the performances of two different student groups.

4.3 Example

Reconsider Example *4.2* assuming further that the same six teachers marked also the papers of a second group of five students (the substitutes of the previous group) examined on the same test. The overall performance of the second group was assessed as in Example *4.2* using TPFNs and the

corresponding mean value was found to be equal to **S'** = (47.8, 65.3, 78.1, 85.9). Compare the performances of the two student groups.
For this, applying Proposition 3.2 one finds that the x-coordinate of the COC of the trapezoid constituting the graph of the TpFN **S** is equal to $X = \frac{79^2+(86.6)^2-(64.2)^2-47^2+79*86.6-47*64.2}{3(79+86.6-47-64.2)} \approx$ 68.84.

In the same way one finds that the x-coordinate of the graph of **S'** is approximately equal to 68.13. Therefore, using the same argument as at the end of Example *4.1*, one finds that the first group demonstrates a better overall performance.

4.4 Remarks

1. It is practically difficult to use the TpFNs in Example *4.1* for assessing the performance of the two Departments, because, due to the great number of students, laborious calculations are needed for this case.

2. Working as in Example *4.3* one can defuzzify the TpFNs S_i, i=1, 2, 3, 4, 5 of Example *4.2* corresponding to the five students of the first group. In this way it becomes possible to compare the individual performance of any two students, in contrast to our method presented in [13] and the equivalent to it method of A. Jones [2] that define a partial order only on the individual student performances.

3. The use of TFNs A, B, C, D and F for the individual student assessment cannot guarantee the definition of a total order on their performances, since each of these TFNs characterizes in general the individual performances of more than one student. This is a good reason to prefer, whenever it is practically possible, the use of TpFNs instead of the TFNs for assessing PS skills.

5 Conclusion

In the present paper we used a combination of TFNs/TpFNs and of the COG defuzzification technique as a tool for assessing student PS skills. The main advantage of the use of the TpFNs for student assessment is that in case of individual assessment it is sufficient for comparing the performances of all students, in contrast to the TFNs and the alternative fuzzy assessment methods that we have applied in earlier works, which define a partial order only on their individual performances. However, in case of group assessment the TFNs/TpFNs approach initially leads to an approximate characterization of the group's overall performance, which is not always sufficient for comparing the performances of two different groups, as our fuzzy assessment methods applied in earlier works do. This is due to the fact that the inequality between TFNs/TpFNs defines on them a relation of partial order only. Therefore, in cases where our fuzzy outputs are not comparable, some extra calculations are needed in order to obtain the required comparison by defuzzifying these outputs. This could be considered as a disadvantage of this approach, although the extra calculations needed are very simple.
Our new method of using TFNs/TpFNs for the assessment of PS skills is of general character, which means that it could be utilized for assessing a great variety of human or machine (e.g. CBR systems) activities. This is one of the main targets of our future research on the subject.

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