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Approximate Solution of Fuzzy Fractional Differential Equations

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Abstract

In this paper we propose a method for computing approximations of solution of fuzzy fractional differential equations using fuzzy variational iteration method. Defining a fuzzy fractional derivative, we verify the utility of the method through two illustrative examples.

Keywords : Fuzzy differential equations; Fractional derivative; Fuzzy variational iteration method; Fuzzy derivative

1 Introduction

Uncertain and incompletely specified systems, can be modelled using fuzzy differential equations. Differential equations which arise in real-word physical problems are often too complicated to solve exactly. And even if an exact solution is obtainable, the required calculations may be too complicated to be practical, or the resulting solution may be difficult to interpret. When some fractional derivatives appear in a differential equation, we will have a fuzzy fractional differential equation. In recent years, fractional differential equations have found applications in many problems in physics and engineering [16, 17]. With uncertainty in initial value of these problems, fuzzy fractional differential equations will also find applications in physics and engineering. Benchohra and Darwish [7] introduced an existence and uniqueness theorem for fuzzy integral equation of fractional order and under some assumptions gave a fuzzy successive iterations which were proved to be uniformly convergent to the unique solution of fuzzy fractional integral equation. In the present paper we will use crisp successive iterations.

In the present paper we propose a method for computing approximations of solution of a Fuzzy Fractional Differential Equation (FFDE) using Variational Iteration Method (VIM). The variational iteration method has been shown [1, 3, 11, 12] to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate crisp solutions in crisp problems. Here we use the advantage of this method to find an approximate solution for FFDE.

The organization of the paper is as follows. In Section 2 we list some basic definitions of fuzzy numbers and fractional derivative and integral. Section 3 and Section 4 contain some theorems about fuzzy initial value problems and FFDE respectively. In Section 5, an approximate solution for FFDE is introduced. Section 6 contains two numerical examples to illustrate the proposed method. Finally, we conclude the paper in Section 7.

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2 Preliminaries

Definition 2.1 *We write $u(t)$, a number in $[0,1]$, for the membership function of u evaluated at t. An α-level of u written $[u]^\alpha$ is defined as $\{t|u(t) \geq \alpha\}$, for $0 < \alpha \leq 1$. We separately define $[u]^0$ as the closure of the union of all the $[u]^\alpha$, $0 < \alpha \leq 1$.*

Let $\mathcal{E}$ denote the class of fuzzy sets on the real line. The parametric form of a fuzzy number can be defined as follows. According to the representation theorem for fuzzy numbers or intervals (see [9]), we use $\alpha-$level setting to define a fuzzy number or interval.

Definition 2.2 *A fuzzy number u is completely determined by any pair $u = (u_1, u_2)$ of functions $u_1, u_2 : [0,1] \to \mathbb{R}$, defining the end-points of the α-levels, satisfying the three conditions:*

(i) $u_1 : \alpha \longmapsto u_1(\alpha) \in \mathbb{R}$ is a bounded monotonic increasing (nondecreasing) left-continuous function $\forall \alpha \in]0,1]$ and right-continuous for $\alpha = 0$;

(ii) $u_2 : \alpha \longmapsto u_2(\alpha) \in \mathbb{R}$ is a bounded monotonic decreasing (nonincreasing) left-continuous function $\forall \alpha \in]0,1]$ and right continuous for $\alpha = 0$;

(iii) $u_1(\alpha) \leq u_2(\alpha)$ $\forall \alpha \in [0,1]$.

If $u_1(1) < u_2(1)$ we have a fuzzy interval and if $u_1(1) = u_2(1)$ we have a fuzzy number. A fuzzy number u is said to be normal if there exist a $t_0 \in \mathbb{R}$ such that $u(t_0) = 1$. We will then consider fuzzy numbers of normal, upper semicontinuous form and we assume that the support $[u_1(0), u_2(0)]$ of u is compact (closed and bounded). The notation $[u]^\alpha = [u_1(\alpha), u_2(\alpha)]$, $\alpha \in [0,1]$ denotes explicitly the α-levels of u.

Let $P_k(\mathbb{R})$ denote the family of all nonempty closed and bounded intervals in $\mathbb{R}$ and define the addition and scalar multiplication in $P_k(\mathbb{R})$ as usual.

Definition 2.3 *If $u, v \in \mathcal{E}$ then for $\alpha \in (0,1]$, $[u \oplus v]^\alpha = [u_1(\alpha) + v_1(\alpha), u_2(\alpha) + v_2(\alpha)]$. If there is such a $w \in \mathcal{E}$ that $u = v \oplus w$, then w is the Hukuhara difference of u and v denoted by $w = u \ominus v$.*

Define $D : \mathcal{E} \times \mathcal{E} \longrightarrow [0, \infty)$ by the equation,

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha),$$

where d is the Hausdorff metric defined in $P_k(\mathbb{R})$. Then, it is easy to see that D is a metric in $\mathcal{E}$ and has the following properties [19, 21]

(1) $(\mathcal{E}, D)$ is a complete metric space;

(2) $D(u \oplus w, v \oplus w) = D(u,v)$ for all $u, v, w \in \mathcal{E}$;

(3) $D(ku, kv) = |k|D(u,v)$

for all $u, v \in \mathcal{E}$ and $k \in \mathbb{R}$. The following theorems and definitions are given in [14, 19]. Let $I = [a,b] \subset \mathbb{R}$ be a compact interval.

Definition 2.4 *A mapping $F : I \to \mathcal{E}$ is strongly measurable if for all α,$(0 < \alpha \leq 1)$, $F_\alpha(t)$ is Lebesgue measurable for any $t \in I$, where $F_\alpha(t) = [F(t)]^\alpha$.*

Definition 2.5 *A mapping $F : I \to \mathcal{E}$ is called levelwise continuous at $t_0 \in I$ if all its α-levels $F_\alpha(t) = [F(t)]^\alpha$ are continuous at $t = t_0$ with respect to the metric d. And F is called integrably bounded if there exist an integrable function h such that $|x| \leqslant h(t)$ for all $x \in [F(t)]^0$.*

We define $F_\alpha(t) = [F(t)]^\alpha = [F_1(t,\alpha), F_2(t,\alpha]$ as α-levels of the mapping $F : I \to \mathcal{E}$. A function $f : I \to \mathbb{R}$ is said to be a measurable selection of $F_\alpha(t)$ if $f(t)$ is measurable and $F_1(t,\alpha) \leq f(t) \leq F_2(t,\alpha)$ for all $t \in I$.

Definition 2.6 *Let $F : I \to \mathcal{E}$. The integral of F over I is defined levelwise by the equation $[\int_I F(t)dt]^\alpha = \int_I F_\alpha(t)dt = \{\int_I f(t)dt | f : I \to \mathbb{R}$ is a measurable selection for $F_\alpha\}$, for all $0 < \alpha \leqslant 1$.*

Theorem 2.1 *If $F : I \to \mathcal{E}$ is strongly measurable and integrably bounded, then F is integrable.*

Definition 2.7 *Reimann-Liouville's fractional derivative and fractional integral of order r $(0 < r < 1)$ for $u(t) : \mathbb{R} \to \mathbb{R}$ are defined as*

$$u^{(r)} = \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} u(s) ds \qquad (2.1)$$

and

$$I^r u(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} u(s) ds. \qquad (2.2)$$

For instance, when $\lambda > -1$ we have

$$\frac{d^r}{dt^r}(t^\lambda) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-r)}t^{\lambda-r} \qquad (2.3)$$

and

$$I^r(t^\lambda) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+r)}t^{\lambda+r}. \qquad (2.4)$$

3 Fuzzy initial value problem

Until now, there have been several methods to deal with the fuzzy initial value problem. Consider the fuzzy initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0 \in \mathcal{E}. \qquad (3.5)$$

There are many suggestions to define a fuzzy derivative and in consequence, to study Eq. (3.5), see for instance [2, 4, 5, 8, 13, 14, 15, 18].

Definition 3.1 *A function $F : [a, b] \to \mathcal{E}$ is differentiable at a point $t_0 \in (a, b)$, if there is such an element $F'(t_0) \in \mathcal{E}$, that the limits*

(I) $\lim_{h\to 0^+} \frac{F(t_0+h)\ominus F(t_0)}{h}$, $\lim_{h\to 0^+} \frac{F(t_0)\ \ominus\ F(t_0-h)}{h}$

or

(II) $\lim_{h\to 0^-} \frac{F(t_0+h)\ominus F(t_0)}{h}$, $\lim_{h\to 0^-} \frac{F(t_0)\ominus F(t_0-h)}{h}$

exist and are equal to $F'(t_0)$. Here the limits are taken in the metric space $(\mathcal{E}, D)$.

In the next section we will define more general type of differentiability for fuzzy fractional differential equations. Now, for $F : [a, b] \to \mathcal{E}$, we easily obtain the following result:

Theorem 3.1 *[13]. Let $F : [a, b] \to \mathcal{E}$ be differentiable and denote $[F(t)]^\alpha = [F_1(t, \alpha), F_2(t, \alpha)]$. Then*

(i) If F is differentiable of type (I), then F_1 and F_2 are differentiable and

$$[F'(t)]^\alpha = [F_1'(t, \alpha), F_2'(t, \alpha)],$$

(ii) If F is differentiable of type (II), then F_1 and F_2 are differentiable and

$$[F'(t)]^\alpha = [F_2'(t, \alpha), F_1'(t, \alpha)].$$

Theorem 3.2 *[20]. (Stacking Theorem) Let $[u_1(\alpha), u_2(\alpha)]$, be a given family of non-empty intervals. If*

(i) $[u_1(\alpha), u_2(\alpha)] \supset [u_1(\beta), u_2(\beta)]$ for $0 < \alpha \leq \beta$

and

(ii) $[\lim_{n\to\infty} u_1(\alpha_n), \lim_{n\to\infty} u_2(\alpha_n)] = [u_1(\alpha), u_2(\alpha)]$

whenever $\{\alpha_n\}$ is a nondecreasing sequence converging to $\alpha \in (0, 1]$, then the family $[u_1(\alpha), u_2(\alpha)], 0 < \alpha \leq 1$, represents the $\alpha-$level sets of a fuzzy number u in $\mathcal{E}$. Conversely, if $[u_1(\alpha), u_2(\alpha)], 0 < \alpha \leq 1$, are the $\alpha-$level sets of a fuzzy number u in $\mathcal{E}$, then the conditions (i) and (ii) hold true.

This gives us a procedure to solve the fuzzy differential equation

$$x'(t) = f(t, x(t)), \quad x(0) = x_0.$$

Denote $[x(t)]^\alpha = [x_1(t, \alpha), x_2(t, \alpha)]$, $[x_0]^\alpha = [x_1^{0,\alpha}, x_2^{0,\alpha}]$ and $[f(t, x(t))]^\alpha = [f_1(t, x_1(t, \alpha), x_2(t, \alpha)), f_2(t, x_1(t, \alpha), x_2(t, \alpha))]$ and proceed as follows:

(i) Solve the differential system for x_1 and x_2

$$\begin{cases} x_1'(t, \alpha) = f_1(t, x_1(t, \alpha), x_2(t, \alpha)) \\ x_2'(t, \alpha) = f_2(t, x_1(t, \alpha), x_2(t, \alpha)). \end{cases} \qquad (3.6)$$

with the initial valuse as: $x_1(0, \alpha) = x_1^{0,\alpha}$ and $x_2(0, \alpha) = x_2^{0,\alpha}$.

(ii) Ensure that $[x_1(t, \alpha), x_2(t, \alpha)]$ and $[x_1'(t, \alpha), x_2'(t, \alpha)]$ are valid level sets. Otherwise, solve the differential system

$$\begin{cases} x_1'(t, \alpha) = f_2(t, x_1(t, \alpha), x_2(t, \alpha)) \\ x_2'(t, \alpha) = f_1(t, x_1(t, \alpha), x_2(t, \alpha)). \end{cases} \qquad (3.7)$$

with the initial valuse as: $x_1(0, \alpha) = x_1^{0,\alpha}$ and $x_2(0, \alpha) = x_2^{0,\alpha}$

and ensure that $[x_1(t, \alpha), x_2(t, \alpha)]$ and $[x_2'(t, \alpha), x_1'(t, \alpha)]$ are valid level sets.

(iii) Using the Stacking Theorem, pile up the levels $[x_1(t,\alpha), x_2(t,\alpha)]$ to a fuzzy solution $x(t)$.

We will use the above mentioned procedure under generalized differentiability. Our main goal is using generalized differentiability for fuzzy initial value problems with fractional derivative.

Theorem 3.3 *[20]. Let $F : I \to \mathcal{E}$ be differentiable and assume that the derivative F' is integrable over I. Then for each $s \in I$, we have $F(s) = F(a) + \int_a^s F'(t)dt$.*

If the fuzzy function $F(t)$ is continuous in the metric D, its definite integral exists [9], and also,

$$[\int_I F(t)dt]^\alpha = [\int_I F_1(t,\alpha)dt, \int_I F_2(t,\alpha)dt]$$

where $[F(t)]^\alpha = [F_1(t,\alpha), F_2(t,\alpha)]$.

Theorem 3.4 *[6] Let us consider FIVP (3.5) where $[f(t,x)]^\alpha = [f_1(t,x_1(t,\alpha),x_2(t,\alpha)), f_2(t,x_1(t,\alpha),x_2(t,\alpha))]$, if $f_1(t,x_1(t,\alpha),x_2(t,\alpha))$ and $f_2(t,x_1(t,\alpha),x_2(t,\alpha))$ are equicontinuous and uniformly bounded on any subset. Also, there exists $L > 0$ such that f_1 and f_2 are Lipschitz continuous functions, then the FIVP (3.5) and the system of ODEs (3.6) or (3.7) are equivalent.*

4 Fuzzy fractional differential equation

In Eq. (3.5), we will replace the first derivative of x with the fractional derivative of x. We use the definition of Reimann-Liouville's fractional derivative of a crisp function in Eq. (2.1) to define fuzzy fractional derivative. Let $F : [0,T] \to \mathcal{E}$ be a fuzzy function defined as following

$$F(t) = \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} x(s)ds$$

where $x(s)$ is a fuzzy function of a crisp variable. For $[x(s)]^\alpha = [x_1(s,\alpha), x_2(s,\alpha)]$ since $t - s > 0$ then $F(t)$ can be found levelwise as $[F(t)]^\alpha = [F_1(t,\alpha), F_2(t,\alpha)] = [\frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} x_1(s,\alpha)ds, \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} x_2(s,\alpha)ds]$. Now, we define the fuzzy fractional derivative as follows.

Definition 4.1 *A function $x : [0,T] \to \mathcal{E}$ has fractional derivative of order r at a point $t_0 \in (0,T)$, if there exists such an element $x^{(r)}(t_0) \in \mathcal{E}$, that the limits*

$$(I)\ \lim_{h\to 0^+} \frac{F(t_0+h) \ominus F(t_0)}{h}\ ,$$

$$\lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0-h)}{h}$$

or

$$(II)\ \lim_{h\to 0^-} \frac{F(t_0+h) \ominus F(t_0)}{h}\ ,$$

$$\lim_{h\to 0^-} \frac{F(t_0) \ominus F(t_0-h)}{h}$$

exist and are equal to $x^{(r)}(t_0)$, where $F(t) = \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} x(s)ds$. Here the limits are taken in the metric space $(\mathcal{E}, D)$.

Consider the following initial value problem with fractional derivative

$$\begin{cases} x^{(r)}(t) = f(x,t), & t \in [t_0,T] \\ x(t_0) = x_0 \in \mathcal{E}, & t_0 \in [0,T) \end{cases} \tag{4.8}$$

In general, it is too difficult to find an exact analytical solution for (4.8), so we will try to find an approximate analytical solution.

Theorem 4.1 *Let $0 < r < 1$ and $x : [a,b] \to \mathcal{E}$ be a fuzzy function with $[x(t)]^\alpha = [x_1(t,\alpha), x_2(t,\alpha)]$. Then*

(i) If x has fractional derivative of type (I), then x_1 and x_2 have fractional derivative and

$$[x^{(r)}(t)]^\alpha = [x_1^{(r)}(t,\alpha), x_2^{(r)}(t,\alpha)]$$

(ii) If x has fractional derivative of type (II), then x_1 and x_2 have fractional derivative and

$$[x^{(r)}(t)]^\alpha = [x_2^{(r)}(t,\alpha), x_1^{(r)}(t,\alpha)]$$

Proof. We prove part (i) and the same proof can be used for part (ii). Since $0 \leqslant s \leqslant t$ and $[x(t)]^\alpha = [x_1(t,\alpha), x_2(t,\alpha)]$ then $[(t-s)^{-r}x(s)]^\alpha = [(t-s)^{-r}x_1(s,\alpha), (t-s)^{-r}x_2(s,\alpha)]$. Since $0 < r < 1$, then $\Gamma(1-r) > 0$, therefore $[\frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r}x(s)ds]^\alpha = [\frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r}x_1(s,\alpha)ds, \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r}x_2(s,\alpha)ds]$ so

$$[F(t)]^\alpha = [F_1(t,\alpha), F_2(t,\alpha)].$$

Now, by the hypothesis of (i) the limits in case (I) of Definition 4.1 exist and then

$$[F'(t)]^\alpha = [F_1'(t,\alpha), F_2'(t,\alpha)]$$

thus we have proved that

$$[x^{(r)}(t)]^\alpha = [x_1^{(r)}(t,\alpha), x_2^{(r)}(t,\alpha)]. \qquad \square$$

Remark 4.1 *When* $[f(t,x)]^\alpha = [f_1(t,x_1(t,\alpha),x_2(t,\alpha)), f_2(t,x_1(t,\alpha),x_2(t,\alpha))]$, *then the Theorem 3.4 shows us a way how to translate the fuzzy fractional differential equations (4.8) into a system of fractional differential equations.*

$$\begin{cases} x_1^{(r)}(t,\alpha) = f_1(t,x_1(t,\alpha),x_2(t,\alpha)) \\ x_2^{(r)}(t,\alpha) = f_2(t,x_1(t,\alpha),x_2(t,\alpha)) \end{cases} \tag{4.9}$$

with the initial values as: $x_1(t_0,\alpha) = x_1^{0,\alpha}$ and $x_2(t_0,\alpha) = x_2^{0,\alpha}$, when part (i) of Theorem 4.1 holds and

$$\begin{cases} x_1^{(r)}(t,\alpha) = f_2(t,x_1(t,\alpha),x_2(t,\alpha)) \\ x_2^{(r)}(t,\alpha) = f_1(t,x_1(t,\alpha),x_2(t,\alpha)) \end{cases} \tag{4.10}$$

with the initial values as: $x_1(t_0,\alpha) = x_1^{0,\alpha}$ and $x_2(t_0,\alpha) = x_2^{0,\alpha}$, when part (ii) of Theorem 4.1 holds. In these equations $[x(t_0)]^\alpha = [x_1^{0,\alpha}, x_2^{0,\alpha}]$ is a fuzzy initial value.

5 Approximate Solution

To illustrate the basic concept of VIM , we consider the following general differential equation

$$Lu + Nu = g(t)$$

where L is a linear operator , N a nonlinear operator and $g(t)$ an inhomogeneous or forcing term. According to the variational iteration method [11], we can construct a correct functional as follows: $u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{Lu_n(s) + N\tilde{u}_n(s) - g(t)\}ds$, where λ is a general Lagrange multiplier which can be identified optimally via the variational theory [12]. The subscript n denotes the n^{th} approximation and $\tilde{u}_n$ considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

In this paper the nonlinear part (Nu), in the Eq. (4.9) are $x_1^{(r)}(t,\alpha)$ and $x_2^{(r)}(t,\alpha)$. As such, the corresponding correctional functional are

$$\begin{cases} x_{1,n+1}^{(r)}(t,\alpha) = x_{1,n}^{(r)}(t,\alpha) + I^r(\lambda_1\{x_{1,n}^{(r)}(t,\alpha) \\ \qquad -f_1(t,x_{1,n}(t,\alpha),x_{2,n}(t,\alpha))\}) \\ x_{2,n+1}^{(r)}(t,\alpha) = x_{2,n}^{(r)}(t,\alpha) + I^r(\lambda_2\{x_{2,n}^{(r)}(t,\alpha) \\ \qquad -f_2(t,x_{1,n}(t,\alpha),x_{2,n}(t,\alpha))\}). \end{cases} \tag{5.11}$$

with the initial values as: $x_1(t_0,\alpha) = x_1^{0,\alpha}$ and $x_2(t_0,\alpha) = x_2^{0,\alpha}$.

Now for the Eq. (4.10), we have

$$\begin{cases} x_{1,n+1}^{(r)}(t,\alpha) = x_{1,n}^{(r)}(t,\alpha) + I^r(\lambda_3\{x_{1,n}^{(r)}(t,\alpha) \\ \qquad -f_2(t,x_{1,n}(t,\alpha),x_{2,n}(t,\alpha))\}) \\ x_{2,n+1}^{(r)}(t,\alpha) = x_{2,n}^{(r)}(t,\alpha) + I^r(\lambda_4\{x_{2,n}^{(r)}(t,\alpha) \\ \qquad -f_1(t,x_{1,n}(t,\alpha),x_{2,n}(t,\alpha))\}). \end{cases} \tag{5.12}$$

with the initial values as: $x_1(t_0,\alpha) = x_1^{0,\alpha}$ and $x_2(t_0,\alpha) = x_2^{0,\alpha}$. To calculate the approximate Lagrange multiplier, we use two integer values $k_1 = ceil(r)$ and $k_2 = floor(r)$, to find μ_1 and μ_2 respectively. For instance, in the first equation of system (4.9) we have

$x_{1,n+1}(t,\alpha) = x_{1,n}(t,\alpha) + \int_0^t(\mu_1\{x_{1,n}^{(k_1)}(t,\alpha) - f_1(t,x_{1,n}(t,\alpha),x_{2,n}(t,\alpha))\})dt$

and

$x_{1,n+1}(t,\alpha) = x_{1,n}(t,\alpha) + \int_0^t(\mu_2\{x_{1,n}^{(k_2)}(t,\alpha) - f_1(t,x_{1,n}(t,\alpha),x_{2,n}(t,\alpha))\})dt$. Finally, we put $\lambda_1 = \beta_1\mu_1 + \beta_2\mu_2$, where β_1 and β_2 are weighted factors with $\beta_1 + \beta_2 = 1$. The same calculation can be used to find λ_2, λ_3 and λ_4.

6 Examples

We use the fractional initial value problem considered in [10] where the initial value is crisp and the homotopy analysis method is used. In this paper the initial value is a fuzzy number and the variational iteration method is used.

Example 6.1 *Consider the following FFDE, with* $t \in [0,1]$

$$\begin{aligned} &x^{(1/2)}(t) = x - t^2 + \frac{2}{\Gamma(2.5)}t^{1.5} \\ &[x(0)]^\alpha = [0, 1/2 - \alpha/2], \quad \alpha \in [0,1]. \end{aligned} \tag{6.13}$$

By Theorem 3.4, the above FFDE is equivalent to the following system

$$\begin{cases} x_1^{(1/2)}(t,\alpha) = f_1(t, x_1(t,\alpha), x_2(t,\alpha)), \\ \qquad\qquad x_1(0,\alpha) = x_1^{0,\alpha} \\ x_2^{(1/2)}(t,\alpha) = f_2(t, x_1(t,\alpha), x_2(t,\alpha)), \\ \qquad\qquad x_2(0,\alpha) = x_2^{0,\alpha}. \end{cases}$$

or

$$\begin{cases} x_1^{(1/2)}(t,\alpha) = f_2(t, x_1(t,\alpha), x_2(t,\alpha)), \\ \qquad\qquad x_1(0,\alpha) = x_1^{0,\alpha} \\ x_2^{(1/2)}(t,\alpha) = f_1(t, x_1(t,\alpha), x_2(t,\alpha)), \\ \qquad\qquad x_2(0,\alpha) = x_2^{0,\alpha}. \end{cases}$$

In each case we use VIM to find approximate solutions for $x_1(t,\alpha) \cong x_{1,m}(t,\alpha)$ and $x_2(t,\alpha) \cong x_{2,m}(t,\alpha)$ for some integer $m > 0$. In this example, we put $\beta_2 = 0$, $\beta_1 = 1$ and with $k_1 = ceil(1/2) = 1$, we have

$x_{1,n+1}(t,\alpha) = x_{1,n}(t,\alpha) + \int_0^t (\mu_1(s)\{x'_{1,n}(s,\alpha) - \tilde{f}_1(s, x_{1,n}(s,\alpha), x_{2,n}(s,\alpha)\})ds$
so
$\delta x_{1,n+1}(t,\alpha) = \delta x_{1,n}(t,\alpha) + \delta\int_0^t (\mu_1(s)\{x'_{1,n}(s,\alpha) - \tilde{f}_1(s, x_{1,n}(s,\alpha), x_{2,n}(s,\alpha)\})ds$

where $\delta\tilde{f}_{1,n}$ denotes restricted variations, i.e. $\delta\tilde{f}_{1,n} = 0$. Making the above correction functionals stationary, we can obtain following stationary conditions:

$$\begin{aligned} &\mu'_1(s) = 0 \\ &1 + \mu_1(s)\mid_{s=t} = 0. \end{aligned} \tag{6.14}$$

Therefore, $\mu_1 = -1$ and then $\lambda_1 = \beta_1\mu_1 + \beta_2\mu_2 = -1$. Analogously, we find $\lambda_2 = -1$. Considering $x_{1,0}(0,\alpha) = 0$ and $x_{2,0}(0,\alpha) = 1/2 - \alpha/2$, and using the following relations

$$\begin{cases} x_{1,n+1}(t,\alpha) = x_{1,n}(t,\alpha) - I^{1/2}(x_{1,n}^{(1/2)}(t,\alpha) \\ \qquad -f_1(t, x_{1,n}(t,\alpha), x_{2,n}(t,\alpha)) \\ x_{2,n+1}(t,\alpha) = x_{2,n}(t,\alpha) - I^{1/2}(x_{2,n}^{(1/2)}(t,\alpha) \\ \qquad -f_2(t, x_{1,n}(t,\alpha), x_{2,n}(t,\alpha)) \end{cases}$$

for $m = n + 1 = 7$, the approximate solutions of the system are obtained by programming through Mathematica software package with high accuracy as

$x_{1,7}(t,\alpha) = 0.9999999999999998t^2 - 0.006947211803185456t^{5.5}$

and

$x_{2,7}(t,\alpha) = t^2 + 1.2528 \times 10^{-16}t^{2.5} + 6.9389 \times 10^{-17}t^3 + 0.04298587303221002t^{3.5} - 0.04298587303221002\alpha t^{3.5} + 1.7347 \times 10^{-17}t^4 + 7.9540 \times 10^{-18}t^{4.5} - 0.006947211803185456t^{5.5}$.

Since $\frac{\partial}{\partial\alpha}(x_{1,7}(t,\alpha)) \geq 0$ and $\frac{\partial}{\partial\alpha}(x_{2,7}(t,\alpha)) < 0$ for each $0 < \alpha < 1$ and $t > 0$, also $x_{1,7}(t,\alpha) < x_{2,7}(t,\alpha)$ because, $x_{2,7}(t,\alpha) - x_{1,7}(t,\alpha) = 1.2527525318167949 \times 10^{-16}t^{2.5} + 6.938893903907227 \times 10^{-17}t^3 + 0.04298587303221002(1-\alpha)t^{3.5} + 1.7347234759768065 \times 10^{-17}t^4 + 7.95398432899552 \times 10^{-18}t^{4.5} > 0$ for all $t > 0$. Therefore, $[x_{1,7}(t,\alpha), x_{2,7}(t,\alpha)]$ are $\alpha-$levels of a fuzzy number which is the approximate solution of the problem.

Example 6.2 *Consider the following FFDE, with $t \in [1,2]$*

$$\begin{aligned} &x^{(1/2)}(t) = x - ln(t) + \frac{ln(4t)}{\sqrt{\pi t}} \\ &[x(1)]^\alpha = [\alpha/3 - 1/3, 1/3 - \alpha/3], \quad \alpha \in [0,1]. \end{aligned} \tag{6.15}$$

By the same calculation as Example 6.1, the approximated solution of (6.15) is obtained as

$x_{1,7}(t,\alpha) = -2.5055 \times 10^{-16} - 2.0583 \times 10^{-16}t^{0.5} - 1.3953 \times 10^{-16}t - 8.1545 \times 10^{-17}t^{1.5} - 4.2008 \times 10^{-17}t^2 - 1.9255 \times 10^{-17}t^{2.5} + 2.6193 \times 10^{-17}t^3 + 0.14037064847641714t^{3.5} + 0.028657248688140003\alpha t^{3.5} + 0.9999999999999999ln(t) - 1.2528 \times 10^{-16}t^{0.5}ln(t) - 1.1102 \times 10^{-16}tln(t) - 8.3517 \times 10^{-17}t^{1.5}ln(t) - 5.5511 \times 10^{-17}t^2ln(t) - 3.3407 \times 10^{-17}t^{2.5}ln(t) - 1.8504 \times 10^{-17}t^3ln(t) - 0.0860t^{3.5}ln(t)$

and

$x_{2,7}(t,\alpha) = -2.5055 \times 10^{-16} - 2.0583 \times 10^{-16}t^{0.5} - 1.3953 \times 10^{-16}t - 8.1545 \times 10^{-17}t^{1.5} - 4.2008 \times 10^{-17}t^2 - 1.9255 \times 10^{-17}t^{2.5} + 2.6193 \times 10^{-17}t^3 + 0.19768514585269714t^{3.5} - 0.028657248688140003\alpha t^{3.5} + 0.9999999999999999ln(t) - 1.2528 \times 10^{-16}t^{0.5}ln(t) - 1.1102 \times 10^{-16}tln(t) - 8.3517 \times 10^{-17}t^{1.5}ln(t) - 5.5511 \times 10^{-17}t^2ln(t) - 3.3407 \times 10^{-17}t^{2.5}ln(t) - 1.8504 \times 10^{-17}t^3ln(t) -$

$0.0860t^{3.5}ln(t)$

Since $\frac{\partial}{\partial\alpha}(x_{1,7}(t,\alpha)) > 0$ and $\frac{\partial}{\partial\alpha}(x_{2,7}(t,\alpha)) < 0$ for each $0 < \alpha < 1$ and $t > 0$, also $x_{1,7}(t,\alpha) < x_{2,7}(t,\alpha)$, because $x_{2,7}(t,\alpha) - x_{1,7}(t,\alpha) = .05731449737628(1-\alpha)t^{3.5} > 0$. Therefore, $[x_{1,7}(t,\alpha), x_{2,7}(t,\alpha)]$ are $\alpha-$levels of a fuzzy number which is the approximate solution of the problem.

Because of the symmetry of initial value, there are some identical coefficients in $x_{1,7}$ and $x_{2,7}$. To emphasize the importance of some coefficients and also differences of $x_{1,7}$ and $x_{2,7}$ we use more digits for some coefficients.

7 Conclusion

Fuzzy fractional differential equations have been introduced via general differentiability. Variational iteration method has been applied to obtain approximate fuzzy solution. Although the example given in this paper is a fuzzy fractional differential equation, it might also be applicable to fuzzy fractional partial differential equations.

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